Heat Loss of Circular Electric Waves in Helix Waveguides*

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Summary—This paper presents a theoretical calculation of the eddy current losses of circular electric waves in a closely-wound helix waveguide. The wire diameter is assumed large compared to the skin depth, but small compared to the guide diameter and the operating wavelength, so that the fields near the wire are quasi-static and may be determined by conformal mapping.

When the wires are in contact, the waveguide wall is effectively a metal surface with grooves of semicircular cross section, the current flow being parallel to the direction of the grooves. The power loss for this case is computed to be about 8.5 per cent higher than in a waveguide with smooth metal walls. When the wires are not in contact, the wall is treated as a grating of parallel, round wires. The increase in power loss over a smooth surface is approximately 22.5 per cent when the wires are separated by a distance equal to their diameter.

I. Introduction

It is well established experimentally[1] that circular electric waves can be propagated in a closely-wound helix waveguide of small round wires, with total losses only slightly higher than the losses in a conventional round waveguide of the same size. In practice the wire diameter will be large compared to the eddy current skin depth, but small compared to the diameter of the guide and the operating wavelength. Under these assumptions, Marcatili[2] has recently used an inverse technique to estimate the increase in eddy current loss in the special case when the wires of the helix are touching, so that the waveguide wall is effectively a metal surface with grooves of semicircular cross section. It is the purpose of this paper to show that the quasi-static problem associated with Marcatili's problem can be solved exactly by conformal mapping, leading to an approximate calculation of the heat loss, and that the eddy current loss can also be calculated approximately when the wires of the helix are not in contact with one another.

II. Boundary With Semicircular Grooves

For a circular electric wave in an ordinary round waveguide the wall currents are purely circumferential, the electric field at the wall is essentially zero, and the magnetic field is in the axial direction. Since helix waveguides are wound with as small a pitch as possible, we may regard the magnetic field at the wall as perpendicular to the wires and the current as parallel to the wires. Since the wires are small compared to the guide radius and the wavelength, we may neglect curvature effects and propagation effects, and consider the two-dimensional problem of a quasi-static magnetic field bounded on one side by a grating of perfectly conducting round wires, and uniform at great distances on the other side. The basic assumption here is that the magnetic field outside the metal is essentially that which would exist outside a smooth perfectly conducting guide wall. The power dissipated by eddy currents is then approximately proportional to the integral of the square of the tangential magnetic field over the conducting surfaces.

A two-dimensional quasi-static magnetic field $He^{i\omega t}$ in the x-y plane may be derived from a complex potential function[3] $W(z)$ by

$$H = \text{grad}\,(\text{Re}\,W), \tag{1}$$

if W is an analytic function of the complex variable $z=x+iy$. The condition that the normal component of H vanishes on a perfectly conducting boundary is satisfied if Im W is constant on such a boundary.

Thus in the case of the periodic semicircular grooves the problem becomes one of finding an analytic function W whose imaginary part is constant on the conducting boundary of Fig. 1 and is such that $W \sim H_0 z$, as $y \rightarrow \infty$, for then, from (1), $H_x \rightarrow H_0$ and $H_y \rightarrow 0$, so that at a great distance from the conducting surface there is a magnetic field of constant strength in the direction indicated in Fig. 1.

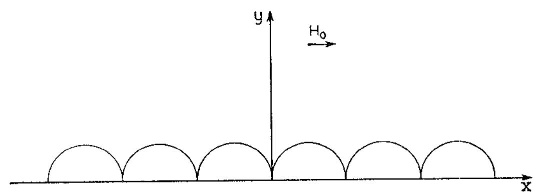

Fig. 1—Cross section of semicircular grooves.

Because of the periodicity of the shape of the conducting surface, it is sufficient to consider just a strip in the z plane as shaded in Fig. 2(a). It is shown in Appendix I that by means of the conformal transformation[4] which

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[1] S. P. Morgan and J. A. Young, "Helix waveguide," *Bell Sys. Tech. J.*, vol. 35, pp. 1347–1384; November, 1956.

[2] E. A. Marcatili, "Heat loss in grooved metallic surface," PROC. IRE, vol. 46, pp. 1134–1139; August, 1957.

[3] S. P. Morgan, "Effect of surface roughness on eddy current losses," *J. Appl. Phys.*, vol. 20, pp. 352–362; April, 1949.

[4] Z. Nehari, "Conformal Mapping," McGraw-Hill Book Co., Inc., New York, N. Y.; 1952.

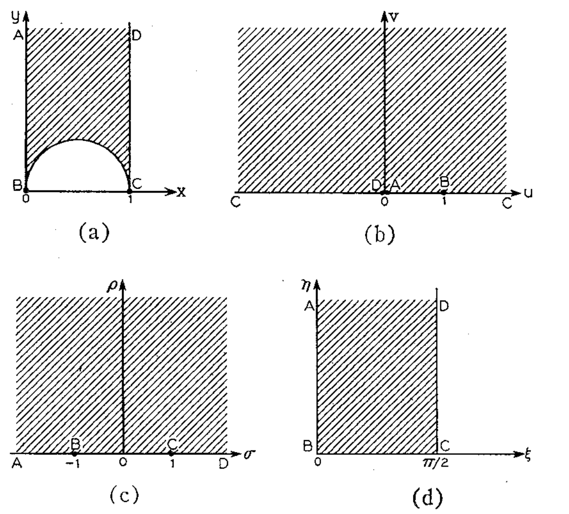

Fig. 2—Conformal transformation from grooved to flat strip. (a) z plane, (b) w plane, (c) τ plane, (d) ζ plane.

maps the upper half-plane into this strip and some elementary transformations, the flat strip $ABCD$ in the $\zeta=\xi+i\eta$ plane, Fig. 2(d), is mapped into the strip in the z plane by the conformal transformation,

$$z=\frac{iK(i\tan\zeta)}{K(\sec\zeta)},\tag{2}$$

where $K(w)$ is the complete elliptic integral of the first kind[5] and modulus w.

In the ζ plane

$$W=\frac{2H_0}{\pi}\zeta,\tag{3}$$

for this makes $\operatorname{Im} W=0$ on $\eta=0$, and hence on the semicircle BC of the conducting surface and further, since the strip in the z plane is of unit width whilst that in the ζ plane is of width $\pi/2$, $W\sim H_0z$ as $y\rightarrow\infty$.

Now the power dissipated by eddy currents in a surface S of conductivity g, whose radius of curvature is large compared to the skin depth δ, is given approximately[3] by

$$P=\frac{1}{2g\delta}\int_S|H_t|^2ds=\frac{1}{2g\delta}\int_S\frac{dW}{dz}\frac{d\overline{W}}{d\bar{z}}ds,\tag{4}$$

where H_t is the amplitude of the tangential component of the magnetic field on the conducting surface, and the second expression follows by using the quasi-static magnetic field given by (1) and the Cauchy-Riemann relations.[4]

For a plane conducting surface, $W=H_0z$. Hence, from (3) and (4) the ratio of the power dissipated in the semicircular groove of unit diameter to that dissipated in a plane strip is

$$\frac{P}{P_0}=\frac{4}{\pi^2}\int_B^C\frac{d\zeta}{dz}\frac{d\bar{\zeta}}{d\bar{z}}ds,\tag{5}$$

where the contour in the z plane is the semicircle in Fig. 2(a). From the relation between z and ζ given in (2) it is shown in Appendix I that

$$\begin{aligned}\frac{P}{P_0}&=\frac{8}{\pi^3}\int_0^1[K(k)]^2dc\\&=\frac{8}{\pi^3}\left[2\int_0^1(1-c)[K(k)]^2dc+1\right],\end{aligned}\tag{6}$$

where $c=k^2$ and $K(k)$ is the complete elliptic integral of the first kind and modulus k. Numerical integration gives

$$\frac{P}{P_0}=1.085.\tag{7}$$

This may be compared with Marcatili's estimate, $P/P_0=1.09\pm0.01$.

We now determine the position of the effective smooth waveguide wall, which enables the phase constant of the propagating wave to be correctly calculated. In Appendix I it is shown that $\zeta\sim(\pi z/2)-i\log 2$, as $y=\operatorname{Im} z\rightarrow+\infty$. Hence, from (3),

$$W\sim H_0\left(z-\frac{2i}{\pi}\log 2\right)=W_0,\text{ as } y\rightarrow+\infty.\tag{8}$$

Thus $\operatorname{Im} W_0=0$ on $y=2/\pi\log 2=d$, which is therefore the position of the effective smooth waveguide wall. Since the wire radius is here supposed to be $\frac{1}{2}$, the ratio D of the displacement of the effective smooth waveguide wall from the trough of the groove to the radius of the grove cross section is

$$D=2d=\frac{4}{\pi}\log 2=0.88.\tag{9}$$

III. Boundary of Cylindrical Wires

Turning now to the case when the pitch of the helix is greater than the wire diameter we are concerned with the plane grating of parallel cylindrical wires, as shown in Fig. 3. There is supposed to be an impressed magnetic field $H_0e^{i\omega t}$ parallel to and on one side of the plane containing the axes of the wires and perpendicular to the wires. There will be leakage through the wires but to the order of the quasi-static approximation this may be neglected. (The circular electric wave can be resolved into conical waves which are reflected on the waveguide walls. In the rectified model this means plane waves incident on the grating and from the equivalent circuit[6] for this problem it follows that the grating acts as a short-circuit in the limit as the ratio of pitch to

[5] P. F. Byrd and M. D. Friedman, "Handbook of Elliptic Integrals for Engineers and Physicists," Springer-Verlag, Berlin, Germany; 1954.

[6] N. Marcuvitz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., M.I.T. Rad. Labs. Ser., vol. 10, p. 286; 1951.

wavelength tends to zero.) The actual leakage is soaked up by a lossy external sheath.

Thus, on the side of the impressed field, the quasi-static magnetic field is of constant strength H_0 at a great distance from the wires, whilst on the other side, the field vanishes at a great distance from the wires. The problem is then one of finding an analytic function W whose imaginary part is constant on the conducting cylinders and is such that $W \to 0$ as $x \to -\infty$ and $W \sim iH_0 z$, as $x \to +\infty$ so that, from (1), $H_x \to 0$ and $H_y \to -H_0$. But Smythe[7] gives the solution to the equivalent electrostatic problem wherein the surfaces $\operatorname{Im} W = 0$ are circular cylinders within two per cent provided that $b \geq 2c$ in Fig. 3, the error decreasing as b/c increases and increasing as b/c decreases. Here $2b$ is the distance between the axes of successive wires and $2c$ is the diameter of the wires both in the direction along the line of axes and in the direction perpendicular to that. As b/c approaches unity, the cylinder becomes a square of side $2c$. The maximum deviation from a circle is quoted later for the three values of the ratio b/c for which numerical results are given.

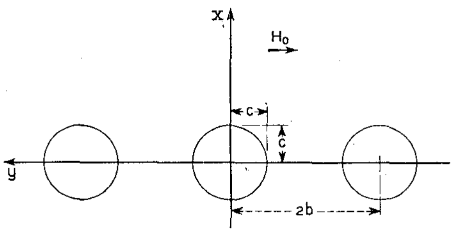

Fig. 3—Cross section of cylindrical wires.

In the present notation, from Smythe,[7]

$$W = \frac{H_0 b}{2\pi}\left(\sin^{-1}(\zeta) + \sin^{-1}\left[\frac{2\zeta + a - 1}{a + 1}\right]\right), \qquad (10)$$

where

$$z = \frac{2b}{\pi(1+\lambda)}\left(\tanh^{-1}\left[\frac{(\zeta - 1)^{1/2}}{(\zeta + a)^{1/2}}\right] + \lambda \tanh^{-1}\left[\frac{(\zeta + 1)^{1/2}}{(\zeta + a)^{1/2}}\right]\right), \qquad (11)$$

and appropriate values are given to the inverse functions and the square roots. The ζ plane is cut along the real axis from -1 to $+\infty$, and from $-a$ to $-\infty$. The quantity λ is the smallest positive root of

$$\sin\left[\frac{\pi c}{2b}(1+\lambda)\right] = \tanh\left[\frac{\pi c}{2b}\left(1 + \frac{1}{\lambda}\right)\right], \qquad (12)$$

and

$$a = \coth^2\left[\frac{\pi c}{2b}\left(1 + \frac{1}{\lambda}\right)\right] + \cot^2\left[\frac{\pi c}{2b}(1+\lambda)\right]. \qquad (13)$$

[7] W. R. Smythe, "Static and Dynamic Electricity," McGraw-Hill Book Co., Inc., New York, N. Y., p. 98; 1950.

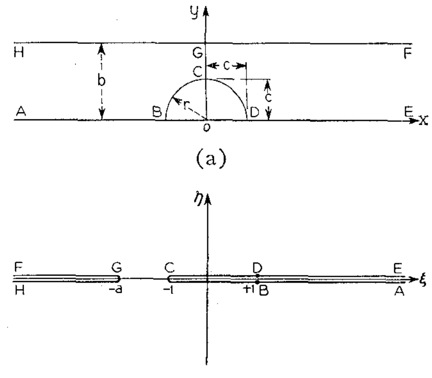

Fig. 4—Conformal transformation of the strip with semicircular intrusion into cut plane. (a) z plane, (b) ζ plane.

These equations arise from the condition that $OC = c = OD$, in Fig. 4(a).

From (4) the ratio of the power dissipated in the parallel grating of cylindrical wires to that dissipated in a plane grating is

$$\frac{P}{P_0} = \frac{1}{bH_0^2}\int_B^D \frac{dW}{dz}\frac{d\overline{W}}{d\bar{z}}\, ds, \qquad (14)$$

since for the plane grating $W = iH_0 z$, $x \geq 0$ and $W = 0$, $x \leq 0$. From (10), (11), and (14) it is shown in Appendix II that

$$\frac{P}{P_0} = (1+\lambda)\left(\frac{[1 - \Lambda_0(\sin^{-1}\lambda, k)]}{\sqrt{1-\lambda^2}} + \cos\left[\frac{\pi c}{2b}(1+\lambda)\right]\cot\left[\frac{\pi c}{2b}(1+\lambda)\right]\frac{K(k)}{\pi\lambda}\right), \qquad (15)$$

where

$$k' = \sqrt{1 - k^2} = \frac{\cos\left[\frac{\pi c}{2b}(1+\lambda)\right]}{\lambda}. \qquad (16)$$

Here $\Lambda_0(\beta, k)$ is Heuman's Lambda function[5] and $K(k)$ is as before. The displacement

$$d = \lim_{x \to +\infty}\left[x - \frac{\operatorname{Im} W}{H_0}\right]$$

of the effective smooth waveguide wall is found in Appendix II, (39), and the ratio $D = d/c$, where c is the axial radius of the wire, is thus

$$D = \frac{b}{\pi c(1+\lambda)}\left((1-\lambda)\log\tanh\left[\frac{\pi c}{2b}\left(1 + \frac{1}{\lambda}\right)\right] + 2\lambda \log\tan\left[\frac{\pi c}{2b}(1+\lambda)\right]\right). \qquad (17)$$

Eq. (12) was solved numerically for three values of b/c and the quantities P/P_0 and D were then calculated from (15) and (17). These values are tabulated below, together with the values r_{max}/c, where r_{max} is the largest value of the radius r of the cylinders forming the grating, as given by Richmond.[8] It is to be expected that as b/c decreases the values of P/P_0 are smaller than those for exactly circular wires and conversely for the values of D.

b/c	P/P_0	D	r_{max}/c
2	1.225	0.64	1.018
3/2	1.140	0.80	1.063
4/3	1.105	0.86	1.11

Since for $b/c=1$ and exactly circular wires the calculated value of P/P_0 is 1.085, (7), the value of P/P_0 for $b/c=4/3$ as given above is about one per cent low. In extrapolating for b/c between 1 and 2 the values calculated for $b/c=1$, 3/2, and 2 should be used. Thus the extrapolated value of P/P_0 for $b/c=6/5$ is 1.10. This compares favorably with the experimental value of 1.13 for the power loss ratio, obtained by J. A. Young of these Laboratories in the case of a helix waveguide with wire diameter 0.0045″ and separation 0.0009.″

IV. Conclusion

An approximate theoretical calculation of heat losses in a metallic waveguide surface with boundary of periodic semicircular grooves and of cylindrical wires has been made, using the magnetic field of a perfect smooth guide. Also calculated was the displacement of the effective smooth waveguide wall from the wire axes. From comparison with experiment it appears that the theory gives quite a good prediction of the heat loss when it is borne in mind that the experimental value will be slightly higher due to leakage into the outer jacket and the presence of a dielectric coating on the wire which presumably tends to concentrate the fields slightly and increase the eddy currents.

It should also be noted that the results apply to any low-loss mode in a helix of finite pitch; *i.e.*, any mode for which, either by accident or design, the wall currents follow the direction of the wires.[1]

Appendix I

We first give the conformal transformation $z=z(\zeta)$ which maps the flat strip in the ζ plane, Fig. 2(d), into the strip with the semicircular groove in the z plane, Fig. 2(a). The successive transformations from the ζ plane are given by[4]

$$\tau = -\cos(2\zeta), \qquad w = \frac{2}{(1-\tau)},$$

$$z = i\frac{K(\sqrt{1-w})}{K(\sqrt{w})}, \tag{18}$$

where

$$K(\sqrt{w}) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-wt^2)}} \tag{19}$$

is the complete elliptic integral of the first kind and modulus $\sqrt{w}$. Combining the transformation of (18) we obtain

$$z = i\frac{K(i\tan\zeta)}{K(\sec\zeta)}. \tag{20}$$

We calculate here the expression for P/P_0 given in (5). On the semicircle in the z plane, Fig. 2(a), setting $\zeta=\xi$, $0\leq\xi\leq(\pi/2)$ in (20), it is found that

$$z = \frac{K}{(K-iK')}, \qquad k = \sin\xi, \tag{21}$$

where K, K' are the complete elliptic integrals of the first kind[5] with complementary moduli k and $k'=\sqrt{1-k^2}$, respectively. From (21), it follows that

$$\frac{dz}{d\xi} = -\frac{i\pi}{2\sin\xi\cos\xi(K-iK')^2}, \tag{22}$$

where use has been made of the identity[5]

$$\frac{d}{dk}\left(\frac{K'}{K}\right) = -\frac{\pi}{2K^2k\,k'^2}. \tag{23}$$

From (5) and (22),

$$\frac{P}{P_0} = \frac{8}{\pi^3}\int_0^{\pi/2}(K^2+K'^2)\sin\xi\cos\xi d\xi$$

$$= \frac{8}{\pi^3}\int_0^1 [K(k)]^2dc; \qquad c = k^2. \tag{24}$$

Since[5]

$$\int_0^1 (2c-1)K^2dc$$

$$= \left[(c-1)(2c-1)K^2 + 2(c-1)KE + E^2\right]_0^1 = 1, \tag{25}$$

(24) may be written

$$\frac{P}{P_0} = \frac{8}{\pi^3}\left(2\int_0^1 (1-c)K^2dc + 1\right). \tag{26}$$

Finally, as $\text{Im}(\zeta)\to+\infty$, $\sec\zeta\to 0$ and $i\tan\zeta\sim -(1-2e^{2i\zeta})$. Hence (5), from (20),

$$\frac{\pi}{2}z \sim i(\log 2 - i\zeta), \text{ as } \text{Im}(\zeta)\to+\infty. \tag{27}$$

Appendix II

We here determine P/P_0 as given by (14), where W and z are given as functions of ζ by (10) and (11). The quantities λ and a are given by (12) and (13) which arise from the condition that $OC=OD=c$ in Fig. 4(a), namely, with

[8] W. R. Richmond, "On the electrostatic field of a plane or circular grating formed of thick round wires," *Proc. Lond. Math. Soc. Ser. 2*, vol. 22, p. 389; 1923.

$$0 < \tan^{-1}\sqrt{\frac{2}{(a-1)}} < \frac{\pi}{2},$$

$$\frac{2b}{\pi(1+\lambda)}\tan^{-1}\sqrt{\frac{2}{(a-1)}} = c = \frac{2b\lambda}{\pi(1+\lambda)}\tanh^{-1}\sqrt{\frac{2}{(a+1)}}. \quad (28)$$

The points A, B, C, D, E, F, G, H in the z plane correspond to the points with the same letters in the ζ plane, Fig. 4(b), which is cut along the real axis from -1 to $+\infty$ and from $-a$ to $-\infty$. From (10), $\text{Im}(W)=0$ for $\zeta=\xi\pm i0$, $-1<\xi\leq 1$; *i.e.*, on the section BCD. In the z plane this section is close to a semicircle if b/c is not close to 1 in which case the section is closer to a square. For $b=2c$ the distance r from the origin O is equal to c within two per cent and for $b=4/3c$ the error is a little over ten per cent.[8]

In differential form (10) and (11) are

$$\frac{dW}{d\zeta} = \frac{iH_0 b}{2\pi}\frac{[(\zeta+1)^{1/2}+(\zeta+a)^{1/2}]}{(\zeta-1)^{1/2}(\zeta+1)^{1/2}(\zeta+a)^{1/2}}, \quad (29)$$

$$\frac{dz}{d\zeta} = \frac{b}{\pi(1+\lambda)}\frac{[(\zeta+1)^{1/2}+\lambda(\zeta-1)^{1/2}]}{(\zeta-1)^{1/2}(\zeta+1)^{1/2}(\zeta+a)^{1/2}}, \quad (30)$$

where appropriate values are to be given to the square roots.

Dividing (29) by (30) we obtain

$$\frac{dW}{dz} = \frac{iH_0(1+\lambda)}{2}\frac{[(\zeta+1)^{1/2}+(\zeta+a)^{1/2}]}{[(\zeta+1)^{1/2}+\lambda(\zeta-1)^{1/2}]}. \quad (31)$$

As $x\to\pm\infty$, $|\zeta|\to\infty$, and assigning the appropriate values to the square roots,

$$\frac{dW}{dz}\to iH_0, \text{ as } x\to+\infty$$
$$\frac{dW}{dz}\to 0, \quad \text{as } x\to-\infty, \quad (32)$$

the difference arising because of the cuts in the ζ plane. This is the desired behavior for W at great distances from the wires.

From (14), (30), and (31),

$$\frac{P}{P_0} = \frac{1}{bH_0^2}\int_{-1}^{1}\left(\left[\frac{dW}{dz}\frac{d\overline{W}}{d\bar{z}}\sqrt{\frac{dz}{d\xi}\frac{d\bar{z}}{d\xi}}\right]_{\zeta=\xi+i0} + \left[\frac{dW}{dz}\frac{d\overline{W}}{d\bar{z}}\sqrt{\frac{dz}{d\xi}\frac{d\bar{z}}{d\xi}}\right]\right)_{\zeta=\xi-i0} d\xi$$

$$= \frac{(1+\lambda)}{2\pi}\cdot\int_{-1}^{1}\frac{[(a+1)+2\xi]}{\sqrt{(1-\xi^2)(a+\xi)[(1+\lambda^2)+(1-\lambda^2)\xi]}}d\xi. \quad (33)$$

By means of standard transformations[5] and after considerable reductions it is found that

$$\frac{P}{P_0} = (1+\lambda)\left(\frac{[1-\Lambda_0(\sin^{-1}\lambda, k)]}{\sqrt{1-\lambda^2}} + \cos\left[\frac{\pi c}{2b}(1+\lambda)\right]\cot\left[\frac{\pi c}{2b}(1+\lambda)\right]\frac{K(k)}{\pi\lambda}\right), \quad (34)$$

where $\Lambda_0(\beta, k)$ is Heuman's Lambda function,[5] λ is the smallest positive root of (12), and

$$(1-k^2) = \frac{\cos^2\left[\frac{\pi c}{2b}(1+\lambda)\right]}{\lambda^2}. \quad (35)$$

Finally, we determine the length

$$d = \lim_{x\to\infty+}\left(x - \frac{\Psi}{H_0}\right), \qquad \Psi = \text{Im } W. \quad (36)$$

Since this is independent of y, it is sufficient to take $y=0$ and consider the behavior as $x\to\infty$ along DE in Fig. 4(a) or, equivalently, as $\xi\to\infty$ on $\zeta=\xi+i0$, Fig. 4(b). From (29), the square roots all being positive on $\zeta=\xi+i0$, $\xi>1$,

$$\Psi = \frac{H_0 b}{2\pi}\left(\log\left[\xi+\sqrt{\xi^2-1}\right] + \log\left[\frac{(2\xi+a-1)+2\sqrt{(\xi+a)(\xi-1)}}{(a+1)}\right]\right), \quad (37)$$

since $\Psi=0$ at $\xi=1$; *i.e.*, on the grating. Also, from (11), for $\zeta=\xi+i0$, $\xi>1$,

$$= x\frac{b}{\pi(1+\lambda)}\left(\log\left[\frac{(2\xi+a-1)+2\sqrt{(\xi+a)(\xi-1)}}{(a+1)}\right]\lambda + \log\left[\frac{(2\xi+a+1)+2\sqrt{(\xi+a)(\xi+1)}}{(a-1)}\right]\right). \quad (38)$$

Thus, from (36)–(38), letting $\xi\to\infty$,

$$d = \frac{b}{2\pi(1+\lambda)}\left((1-\lambda)\log\left[\frac{2}{(a+1)}\right] + 2\lambda\log\left[\frac{2}{(a-1)}\right]\right).$$

$$= \frac{b}{\pi(1+\lambda)}\left((1-\lambda)\log\tanh\left[\frac{\pi c}{2b}\left(1+\frac{1}{\lambda}\right)\right] + 2\lambda\log\tan\left[\frac{\pi c}{2b}(1+\lambda)\right]\right), \quad (39)$$

using (28).

V. Acknowledgment

The writer wishes to thank S. P. Morgan for bringing this problem to his attention and for many helpful discussions during the preparation of this paper.